

# THE SMASH PRODUCT FOR DERIVED CATEGORIES IN STABLE HOMOTOPY THEORY

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**ABSTRACT.** An  $E_1$  (or  $A_\infty$ ) ring spectrum  $R$  has a derived category of modules  $\mathcal{D}_R$ . An  $E_2$  structure on  $R$  endows  $\mathcal{D}_R$  with a monoidal product  $\wedge_R$ . An  $E_3$  structure on  $R$  endows  $\wedge_R$  with a braiding. If the  $E_3$  structure extends to an  $E_4$  structure then the braided monoidal product  $\wedge_R$  is symmetric monoidal.

## INTRODUCTION

Stable homotopy theory is essentially the study of generalized homology and cohomology theories. From its beginning in the work of Spanier and Whitehead on duality in the 1950's and the work of Adams, Atiyah and Hirzebruch, Thom, Quillen, and many others on vector fields, topological  $K$ -theory, and cobordism theory in the 1950's, 1960's, and 1970's, stable homotopy theory has provided powerful tools for studying questions in geometry and topology. Many of algebraic topology's deepest advances and greatest successes have been tied to the development of new cohomology theories and the study of stable phenomena.

Because cohomology theories involve long exact sequences, very few algebraic constructions work without severe flatness hypotheses. Stable homotopy theorists therefore study a refinement (due to Boardman) of the category of cohomology theories, called the “stable category”, whose objects are usually called “spectra”. This category has a “smash product” that captures multiplicative structures on cohomology theories: Roughly speaking, multiplicative cohomology theories tend to be represented by “homotopical ring spectra”, defined in terms of monoids for the smash product. Actions of homotopical ring spectra define “homotopical module spectra”, which represent cohomology theories that are modules over ring theories. Properties of homotopical ring spectra often extend to simplify computations involving homotopical module spectra, and vice-versa.

The stable category with its smash product provides a good context for stable homotopy theory, and the notions of homotopical ring and module spectra suffice for many purposes, as amply demonstrated in the literature since the 1960's. On the other hand, as addressed by May and collaborators by the mid 1970's and as became widely acknowledged by the mid 1980's, certain necessary constructions require a stronger point-set foundation. For example, homotopy ring spectra are the stable analogue of homotopy associative  $H$ -spaces rather than the analogue of topological monoids; because of this, few of the constructions available in the stable category preserve homotopical module spectra.

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The papers [6, 9, 15] rewrote the foundations of stable homotopy theory, providing several categories whose homotopy categories are the stable category but which have symmetric monoidal point-set smash products (before passing to the homotopy category). Current terminology calls the monoids and commutative monoids for these smash products  $S$ -algebras and commutative  $S$ -algebras; these are essentially equivalent to the older notions of  $A_\infty$  and  $E_\infty$  ring spectra, respectively. As a consequence of the modern foundations, for an  $S$ -algebra  $R$ , the category of point-set left (or right)  $R$ -modules has an intrinsic homotopy theory. The homotopy category, usually called the “derived category”, shares most of the structure of the stable category and admits most of the usual constructions in homotopy theory, with the possible exception of those that require an internal smash product.

In general, for an  $S$ -algebra  $R$ , we can form the balanced product “ $\wedge_R$ ” of a right  $R$ -module and a left  $R$ -module as a functor from the derived categories to the stable category

$$\wedge_R: \mathcal{D}_R^{\text{right}} \times \mathcal{D}_R \longrightarrow \mathcal{S}$$

(where  $\mathcal{D}_R$  denotes the derived category of left  $R$ -modules,  $\mathcal{D}_R^{\text{right}}$  denotes the derived category of right  $R$ -modules, and  $\mathcal{S}$  denotes the stable category). As in the case of ordinary rings in algebra, when  $R$  is a commutative  $S$ -algebra, left and right  $R$ -modules are equivalent, and the balanced product lifts to an internal smash product

$$\wedge_R: \mathcal{D}_R \times \mathcal{D}_R \longrightarrow \mathcal{D}_R,$$

which is a closed symmetric monoidal product. Unlike the case of ordinary rings in algebra, ring spectra admit an infinite hierarchy of structures between  $S$ -algebra and commutative  $S$ -algebra, the  $E_n$  hierarchy of Boardman and Vogt [4]. An  $E_1$  ring spectrum is an  $A_\infty$  ring spectrum, is equivalent to an  $S$ -algebra, and has a derived category of left modules. An  $E_\infty$  ring spectrum is equivalent to a commutative  $S$ -algebra and its derived category has a symmetric monoidal product. This paper begins the study of the derived categories of left modules over  $E_n$  ring spectra for  $1 < n < \infty$ . The main theorem is:

**Main Theorem.** *Let  $R$  be an  $E_2$  ring spectrum.*

- (i) *The derived category of left modules  $\mathcal{D}_R$  is equivalent to the derived category of right modules  $\mathcal{D}_R^{\text{right}}$  and has a closed monoidal product  $\wedge_R$  extending the balanced product.*
- (ii) *If  $R$  is an  $E_3$  ring spectrum, then  $\wedge_R$  has a braiding.*
- (iii) *If  $R$  is an  $E_4$  ring spectrum then the braiding is a symmetry, i.e.,  $\mathcal{D}_R$  is a closed symmetric monoidal category.*

As one of the principle interests in constructing  $E_\infty$  structures on ring spectra has been to have a monoidal or symmetric monoidal category of modules, for statements in the derived category, now merely an  $E_2$  or  $E_4$  structure suffices. For example, Maria Basterra and the author have shown that the Brown Peterson spectrum  $BP$  at each prime is an  $E_4$  ring spectrum [3]; it is currently not known whether it is an  $E_\infty$  ring spectrum.

To avoid a point of possible confusion, we emphasize that the derived category  $\mathcal{D}_R$  in the theorem above is the derived category of left modules for  $R$  regarded as an  $A_\infty$  ring spectrum, and not, for example, the derived category of operadic modules for  $R$  regarded as an  $E_n$  ring spectrum. See Section 1 for a review of the precise definition of  $\mathcal{D}_R$ .

The main theorem addresses only the question of derived categories or homotopy categories. In fact, the smash product in the homotopy category derives from a point-set level “lax monoidal product” [11, 3.1.1] or “partial lax monoidal product”, which we outline in Section 5. In lectures on this work dating back to 2004, the author has presented the following general conjecture, converse to the main theorem (in the  $E_2$  case):

**Conjecture.** *Under suitable technical hypotheses, a lax or partial lax monoidal product on a category with structure maps weak equivalences induces an  $E_2$  structure on the derived endomorphism ring spectrum of the unit.*

The previous conjecture generalizes the Deligne Hochschild cohomology conjecture, which is the special case of the monoidal category of  $(A, A)$ -bimodules over a ring (or DG ring or  $S$ -algebra). In this case, the derived endomorphism DG algebra (or ring spectrum) is the (topological) Hochschild cohomology complex. The (affirmed) Deligne conjecture is that this is an  $E_2$  algebra [18].

More generally, the author has advertised the problem of identifying the point-set structure on the category of modules over an  $E_n$  ring spectrum (for  $n > 2$ ), extending the lax monoidal structure. Once identified, a corresponding converse conjecture could be formulated. With the new understanding of quasi-categories that has developed in the time since the author first announced the main theorem, the conjecture above and its generalization to  $E_n$  ring spectra (for all  $n$ ) have become feasible to approach. The author understands that these and related problems have since been solved by Clark Barwick [2] and David Gepner [7]; see also Lurie’s treatment in [14, 2.3.15].

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## 1. OUTLINE AND PRELIMINARIES

Although the constructions in this paper would presumably work in any modern (topological) category of spectra, for definiteness we work in the category of EKMM  $S$ -modules; this allows us to take some technical shortcuts in several places using the fact that all objects are fibrant. For  $E_n$  algebras, we work exclusively with the *little  $n$ -cubes* operads  $\mathfrak{C}_n$  of Boardman and Vogt [4]: An element of  $\mathfrak{C}_n(m)$  consists of  $m$  almost disjoint *sub-cubes* of the unit cube  $[0, 1]^n$ , labelled  $1, \dots, m$ , of the form

$$[x_1^i, y_1^i] \times \cdots \times [x_n^i, y_n^i]$$

(for  $0 \leq x_j^i < y_j^i \leq 1$ , but generally not with equal side lengths  $y_j^i - x_j^i$ : These are affinely embedded sub-cubes, rather than actual geometric sub-cubes). An  $E_n$  algebra in this context is then an  $S$ -module  $R$  together with an action

$$\mathfrak{C}_n(m)_+ \wedge_{\Sigma_m} R^{(m)} \longrightarrow R$$

satisfying the usual properties (where  $R^{(m)} = R \wedge_S \cdots \wedge_S R$ ). As a technical remark for those familiar with  $E_n$  ring spectra in the sense of Lewis and May [12], we note that this is precisely an  $E_n$  ring spectrum  $R$  for the operad  $\mathfrak{C}_n \times \mathfrak{L}$  such that the underlying  $\mathbb{L}$ -spectrum of  $R$  is an  $S$ -module [6, II.1.1]. The usual theory [16] (cf. [6, XII§1, II§4]) shows that any other sort of  $E_n$  ring spectrum is equivalent to one of this type in an essentially unique way.

We denote by  $\mathfrak{A}$  the non- $\Sigma$  operad of little 1-cubes: An element of  $\mathfrak{A}(k)$  is a sequence of  $k$  almost disjoint sub-intervals of the unit interval in order. Then  $\mathfrak{A}(k) \subset \mathfrak{C}_1(k)$ , and as an operad  $\mathfrak{C}_1 \cong \mathfrak{A} \times \Sigma$ ; thus,  $\mathfrak{A}$ -algebras and  $\mathfrak{C}_1$ -algebras coincide. We regard  $\mathfrak{C}_n$ -algebras as  $\mathfrak{A}$ -algebras via the usual inclusion of  $\mathfrak{C}_1$  in  $\mathfrak{C}_n$  (taking a sub-interval  $[x, y]$  to the sub-cube  $[x, y] \times [0, 1]^{n-1}$ ).

For an  $\mathfrak{C}_n$ -algebra  $R$ , we understand a left  $R$ -module to be an operadic left module for  $R$  regarded as an  $\mathfrak{A}$ -algebra. In other words, a left  $R$ -module consists of an  $S$ -module  $M$  and maps of  $S$ -modules

$$\mathfrak{A}(m+1)_+ \wedge R^{(m)} \wedge_S M \longrightarrow M$$

for all  $m$ , satisfying the usual associativity and unit diagrams (as in, for example, [10, I.4.2.(ii)]), reviewed in Section 2. We use  $\mathcal{M}_R$  to denote the category of left  $R$ -modules. For purely formal reasons,  $\mathcal{M}_R$  is a category of modules over an  $S$ -algebra  $U_{\mathfrak{A}}R$  (or just  $UR$ ), the *left module enveloping algebra* of  $R$ , which we review in Section 2. In fact, using the details of the little 1-cubes non- $\Sigma$  operad  $\mathfrak{A}$ , we give a concrete description of  $UR$ . Using that description, we prove the following result on enveloping algebras. This result is a special feature of  $\mathfrak{A}$  not shared by a general  $A_\infty$  operad without additional hypotheses on the  $A_\infty$  algebra  $R$ .

**Theorem 1.1.** *For any  $\mathfrak{A}$ -algebra  $R$ , the canonical map of left  $UR$ -modules  $UR \rightarrow R$  induced by the unit of  $S \rightarrow R$  is a homotopy equivalence of  $S$ -modules.*

We understand the derived category of left  $R$ -modules  $\mathcal{D}_R$  to be the derived category of  $UR$ -modules  $\mathcal{D}_{UR}$  [6, III§2], obtained by formally inverting the weak equivalences. We build the smash product on  $\mathcal{D}_R$  in the Main Theorem by combining a formal construction on  $\mathcal{M}_R$  with some homotopical results.

The formal construction involves the “interchange” property of the operads  $\mathfrak{C}_1$  and  $\mathfrak{C}_{n-1}$  for a  $\mathfrak{C}_n$ -algebra  $R$ . Pairwise cartesian product of sub-cubes defines a map

$$\mathfrak{C}_1(\ell) \times \mathfrak{C}_{n-1}(m) \longrightarrow \mathfrak{C}_n(\ell m)$$

that is a *pairing of operads* [17]. We use this pairing in Section 3 to associate to every element of  $\mathfrak{C}_{n-1}(m)$  a natural map of  $\mathfrak{A}$ -algebras

$$R \wedge_S \cdots \wedge_S R \longrightarrow R$$

and hence a map of  $S$ -algebras  $U(R^{(m)}) \rightarrow UR$ . As a variant of this, for any space  $X$  and map  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$ ,  $UR \wedge X_+$  becomes a  $(UR, U(R^{(m)}))$ -bimodule, and hence defines a functor

$$f_*: \mathcal{M}_{R^{(m)}} \longrightarrow \mathcal{M}_R, \quad f_*M = UR \wedge X_+ \wedge_{U(R^{(m)})} M.$$

The diagonal map  $\mathfrak{A} \rightarrow \mathfrak{A}^m$  defines a map of  $S$ -algebras

$$(1.2) \quad U(R^{(m)}) = U_{\mathfrak{A}}(R^{(m)}) \longrightarrow U_{\mathfrak{A}^m}(R^{(m)}) \cong (UR)^{(m)},$$

which defines a forgetful or pullback functor

$$\mathcal{M}_{(UR)^{(m)}} \longrightarrow \mathcal{M}_{U(R^{(m)})} = \mathcal{M}_{R^{(m)}}.$$

Composing these functors with the smash product over  $S$

$$\mathcal{M}_R \times \cdots \times \mathcal{M}_R = \mathcal{M}_{UR} \times \cdots \times \mathcal{M}_{UR} \longrightarrow \mathcal{M}_{(UR)^{(m)}},$$

we obtain a functor

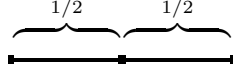
$$(1.3) \quad \Lambda_f: \mathcal{M}_R \times \cdots \times \mathcal{M}_R \longrightarrow \mathcal{M}_R.$$

In other words,

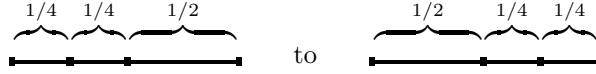
$$\Lambda_f(M_1, \dots, M_m) = UR \wedge X_+ \wedge_{U(R^{(m)})} (M_1 \wedge_S \dots \wedge_S M_m).$$

We call these operations  $E_n$  *interchange operations*.

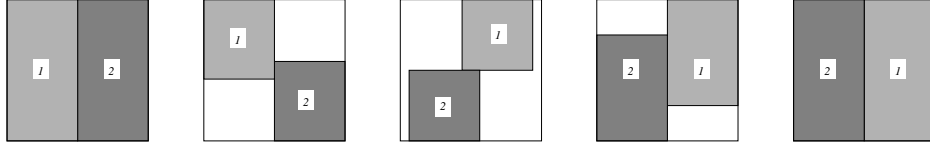
When  $n = 2$ , we use  $X = *$  and  $f$  the element  $\mu = ([0, 1/2], [1/2, 1])$



of  $\mathfrak{A}(2) \subset \mathfrak{C}_1(2)$  to construct a functor  $\Lambda_\mu$  that provides point-set version of the smash product functor for the Main Theorem. We use  $X$  an interval and  $f$  a path  $\alpha$  from



in  $\mathfrak{A}(3) \subset \mathfrak{C}_1(3)$  as a key component of the construction of the associativity isomorphisms (in  $\mathscr{D}_R$ ) for the smash product (see also 1.6 below). We use maps from the pentagonal disk to  $\mathfrak{C}_1$  to establish coherence; see Section 4 for details. For  $n = 3$ , we use a path like the one pictured



in  $\mathfrak{C}_2(2)$  to construct a braiding, and for  $n = 4$ , a null homotopy in  $\mathfrak{C}_3(2)$  of the composition of such paths to prove the symmetry. See Section 4 for details.

The  $E_n$  interchange operations  $\Lambda_f$  do not strictly preserve composition, and this introduces some complications into the formal picture. To illustrate, let  $f: X \rightarrow \mathfrak{C}_1(2)$  and  $g: Y \rightarrow \mathfrak{C}_1(2)$ . We obtain a map  $f \circ_2 g: X \times Y \rightarrow \mathfrak{C}_1(3)$  by operadic composition and hence a functor

$$\Lambda_{f \circ_2 g}: \mathcal{M}_R \times \mathcal{M}_R \times \mathcal{M}_R \longrightarrow \mathcal{M}_R,$$

which is defined by

$$\Lambda_{f \circ_2 g}(L, M, N) = UR \wedge (X \times Y)_+ \wedge_{U(R^{(3)})} (L \wedge_S M \wedge_S N).$$

On the other hand, the composition of operations  $\Lambda_f \circ_2 \Lambda_g$  is the functor

$$\begin{aligned} \Lambda_f(L, \Lambda_g(M, N)) &= UR \wedge X_+ \wedge_{U(R^{(2)})} (L \wedge_S (UR \wedge Y_+ \wedge_{U(R^{(2)})} (M \wedge_S N))) \\ &\cong UR \wedge (X \times Y)_+ \wedge_{UR \wedge_S U(R^{(2)})} (L \wedge_S M \wedge_S N). \end{aligned}$$

Specifically,  $\Lambda_{f \circ_2 g}$  treats the  $(UR)^{(3)}$ -module  $L \wedge_S M \wedge_S N$  as a  $U(R^{(3)})$ -module, while  $\Lambda_f \circ_2 \Lambda_g$  treats it as a  $UR \wedge_S U(R^{(2)})$ -module. A generalization of (1.2) induces a map of  $S$ -algebras from  $U(R^{(3)})$  to  $UR \wedge_S U(R^{(2)})$ , and so induces a natural transformation

$$\Lambda_{f \circ_2 g} \longrightarrow \Lambda_f \circ_2 \Lambda_g.$$

More generally, for a  $\mathfrak{C}_n$ -algebra  $R$ , given a map  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$  and maps  $g_i: Y_i \rightarrow \mathfrak{C}_{n-1}(j_i)$ , we have a natural transformation

$$(1.4) \quad \Lambda_{f \circ (g_1, \dots, g_m)} \longrightarrow \Lambda_f \circ (\Lambda_{g_1}, \dots, \Lambda_{g_m})$$

of functors  $\mathcal{M}_R \times \cdots \times \mathcal{M}_R$  to  $\mathcal{M}_R$ . Although this transformation is not an isomorphism, in Section 3, we show that it is often a weak equivalence.

**Theorem 1.5.** *With notation as above, for cofibrant  $R$ -modules  $M_1, \dots, M_j$  with  $j = j_1 + \cdots + j_m$ , the natural map (1.4)*

$$\Lambda_{f \circ (g_1, \dots, g_m)}(M_1, \dots, M_j) \longrightarrow \Lambda_f(\Lambda_{g_1}(M_1, \dots, M_{j_1}), \dots, \Lambda_{g_m}(M_{j-j_m+1}, \dots, M_j))$$

*is a weak equivalence.*

We apply Theorem 1.5 in Section 4 to construct the coherence isomorphisms in  $\mathcal{D}_R$  for the Main Theorem. For example, for  $\mu \in \mathfrak{C}_1(2)$  and  $\alpha: I \rightarrow \mathfrak{C}_1(3)$  as above, the maps

$$(1.6) \quad \Lambda_\mu \circ_2 \Lambda_\mu \longleftarrow \Lambda_{\mu \circ_2 \mu} \longrightarrow \Lambda_\alpha \longleftarrow \Lambda_{\mu \circ_1 \mu} \longrightarrow \Lambda_\mu \circ_1 \Lambda_\mu$$

induce isomorphisms in  $\mathcal{D}_R$ , which construct the associativity isomorphism for the Main Theorem. See Section 4 for details. To make this work and to use (1.6) to construct an isomorphism of left derived functors, we need to understand composition of the left derived functors of the operations  $\Lambda_f$ . For this, we have the following theorem proved in Section 3.

**Theorem 1.7.** *Let  $R$  be a  $\mathfrak{C}_n$ -algebra,  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$  a map, and  $M_1, \dots, M_m$   $R$ -modules. If  $X$  is homotopy equivalent to a CW complex and  $M_1, \dots, M_n$  are homotopy equivalent to cofibrant  $R$ -modules, then  $\Lambda_f(M_1, \dots, M_m)$  is homotopy equivalent to a cofibrant  $R$ -module.*

This theorem in particular implies that the left derived functor of a composite of  $E_n$  interchange operations is the corresponding composite of derived functors.

**Outline.** In Section 2, we review the left module enveloping algebra and prove Theorem 1.1. In Section 3, we study the homotopy theory of  $R$ -modules and the operations  $\Lambda_f$ ; we prove Theorems 1.5 and 1.7. In Section 4, we apply this theory to prove the Main Theorem. Section 5 discusses the point-set lax monoidal refinement of the constructions that go into the proof of the Main Theorem. Section 5 also discusses the converse conjecture in the introduction and further generalizations of the Deligne conjecture (and their converses).

The final section, Section 6, bears no direct relationship to the Main Theorem, but rather provides a follow-up to the proof of Theorem 1.1 and the concrete description of the left module enveloping algebra  $UR$ . For an  $\mathfrak{A}$ -algebra  $R$ , an alternative concrete construction, like the construction of the Moore loop space, produces an associative algebra  $R_M$  that we call the “Moore algebra”. In Section 6, we construct a natural zigzag of weak equivalences between the left module enveloping algebra  $UR$  and the Moore algebra  $R_M$ . This then relates the categories of  $R$ -modules to  $R_M$ -modules.

## 2. THE LEFT MODULE ENVELOPING ALGEBRA

For an  $\mathfrak{A}$ -algebra  $R$ , a left  $R$ -module consists of an  $S$ -module  $M$  together with action maps

$$\xi_m: \mathfrak{A}(m+1)_+ \wedge R^{(m)} \wedge M \longrightarrow M$$

satisfying the usual conditions. Writing  $\zeta$  for the  $\mathfrak{A}$ -algebra multiplication of  $R$ , these conditions are the associativity diagrams

$$\begin{array}{ccc}
(\mathfrak{A}(m+1) \times (\mathfrak{A}(j_1) \times \cdots \times \mathfrak{A}(j_m) \times \mathfrak{A}(j_{m+1}+1)))_+ \wedge R^{(j)} \wedge_S M & \xrightarrow{\circ \wedge \text{id}} & \mathfrak{A}(j+1)_+ \wedge R^{(j)} \wedge_S M \\
\cong \downarrow & & \downarrow \xi_j \\
\mathfrak{A}(m)_+ \wedge ((\mathfrak{A}(j_1)_+ \wedge R^{(j_1)}) \wedge_S \cdots \wedge_S (\mathfrak{A}(j_m)_+ \wedge R^{(j_m)}) \wedge_S (\mathfrak{A}(j_{m+1}+1)_+ \wedge R^{(j_{m+1})} \wedge_S M)) & & \\
\text{id} \wedge \zeta_{j_1} \wedge \cdots \wedge \zeta_{j_m} \wedge \xi_{j_{m+1}} \downarrow & & \\
\mathfrak{A}(m+1)_+ \wedge R^{(m)} \wedge_S M & \xrightarrow{\xi_m} & M
\end{array}$$

(for  $m, j_1, \dots, j_{m+1} \geq 0$  and  $j = j_1 + \cdots + j_{m+1}$ ) and the unit diagram

$$\begin{array}{ccc}
\{\mathbf{1}\}_+ \wedge R^{(0)} \wedge_S M & \longrightarrow & \mathfrak{A}(1)_+ \wedge R^{(0)} \wedge_S M \\
\cong \downarrow & & \downarrow \xi_1 \\
S^0 \wedge S \wedge_S M & \xrightarrow{\cong} & M,
\end{array}$$

where  $\mathbf{1}$  denotes the identity element of  $\mathfrak{A}(1)$  (the whole sub-interval  $[0, 1]$  of  $[0, 1]$ ). The action maps (as generators), the associativity diagrams (as relations), and the unit diagram (as the unit) implicitly specify an  $S$ -algebra  $UR$  that encodes an  $R$ -module structure. We begin with this construction.

Let  $UR$  be the  $S$ -module formed as the coequalizer of the following diagram:

$$\bigvee_{m, j_1, \dots, j_m} (\mathfrak{A}(m+1) \times (\mathfrak{A}(j_1) \times \cdots \times \mathfrak{A}(j_m)))_+ \wedge R^{(j)} \rightrightarrows \bigvee_m \mathfrak{A}(m+1)_+ \wedge R^{(m)},$$

where one map is induced by the operadic multiplication  $\circ$  and the other by the  $\mathfrak{A}$ -algebra multiplication of  $R$ . The operadic composition of  $\mathfrak{A}$  using the last sub-interval,

$$\mathfrak{A}(m+1) \circ_{m+1} \mathfrak{A}(k+1) \longrightarrow \mathfrak{A}(m+k+1),$$

induces a multiplication map  $UR \wedge_S UR \rightarrow UR$ , and the inclusion of the element  $\mathbf{1}$  in  $\mathfrak{A}(1)$  induces a unit map  $S \rightarrow UR$ . Since the operadic composition is associative and unital,

$$\mathbf{a} \circ_{\ell+1} (\mathbf{b} \circ_{m+1} \mathbf{c}) = (\mathbf{a} \circ_{\ell+1} \mathbf{b}) \circ_{\ell+m+1} \mathbf{c} \quad \text{and} \quad \mathbf{1} \circ_1 \mathbf{a} = \mathbf{a} = \mathbf{a} \circ_{m+1} \mathbf{1},$$

it follows that the multiplication and unit maps make  $UR$  into an associative  $S$ -algebra.

**Definition 2.1.** The  $S$ -algebra  $UR$  is called the *left module enveloping algebra*.

Comparing the universal property defining  $UR$  with the data defining a left  $R$ -module leads to the following proposition (cf. [8, 1.6.6], [10, I.4.10]):

**Proposition 2.2.** *A left  $R$ -module structure on an  $S$ -module determines and is determined by a left  $UR$ -module structure.*

**Convention 2.3.** By slight abuse, we use left  $R$ -modules and left  $UR$ -modules interchangeably. We define the category of left  $R$ -modules  $\mathcal{M}_R$  to be the category of left  $UR$ -modules  $\mathcal{M}_{UR}$ .

The construction of  $UR$  above is purely formal, using none of the specifics of  $\mathfrak{A}$ ; indeed, the analogue of construction makes sense for an arbitrary non- $\Sigma$  operad, and Proposition 2.2 holds in full generality. On the other hand, for the non- $\Sigma$  operad  $\mathfrak{A}$ , the left module enveloping algebras admit a more concrete description, which we now produce.

Let  $D$  denote the subspace of  $\mathfrak{A}(2)$  where the first sub-interval begins at zero and the second sub-interval begins at the same point where the first one ends:

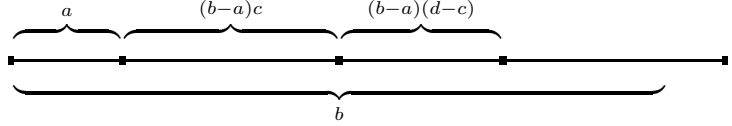
$$D = \{([x_1, y_1], [x_2, y_2]) \in \mathfrak{A}(2) \mid x_1 = 0, y_1 = x_2\}.$$

Let  $\bar{D} = \mathfrak{A}(1)$ ; then dropping the first sub-interval includes  $D$  in  $\bar{D}$  as the subspace of intervals that do not start at 0. Let  $A = AR$  be the  $S$ -module defined by the following pushout diagram.

$$\begin{array}{ccc} D_+ \wedge S & \longrightarrow & D_+ \wedge R \\ \downarrow & & \downarrow \\ \bar{D}_+ \wedge S & \longrightarrow & A \end{array}$$

Intuitively,  $A$  consists of pairs of sub-intervals  $([0, a], [a, b])$ , with the first sub-interval labelled by  $R$ , union sub-intervals  $[0, b]$  labelled by  $S$ .

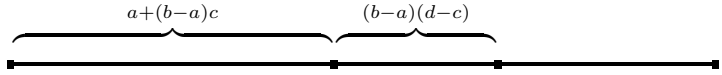
We use  $\circ_2$  to construct an associative multiplication on  $A$  as follows. Given  $\mathbf{a} = ([0, a], [a, b])$  and  $\mathbf{c} = ([0, c], [c, d])$  in  $D$ , then  $\mathbf{a} \circ_2 \mathbf{c}$  “plugs”  $\mathbf{c}$  into the second sub-interval in  $\mathbf{a}$ , producing three sub-intervals,



These define a new element of  $D$  by taking the first sub-interval to be the concatenation of the first two sub-intervals above and taking the second sub-interval to be the remaining (third) sub-interval above. In formulas, this is the pair

$$([0, a + (b - a)c], [a + (b - a)c, a + (b - a)d]),$$

and pictorially is

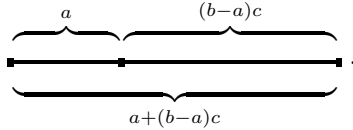


This defines a map  $p: D \times D \rightarrow D$ .

To explain what happens with the  $R \wedge_S R$  factor of  $(D_+ \wedge R)^{(2)}$ , we use the first two sub-intervals in  $\mathbf{a} \circ_2 \mathbf{c}$  to specify a map  $q: D \times D \rightarrow \mathfrak{A}(2)$ . Let  $q$  be the map sending  $(\mathbf{a}, \mathbf{c})$  as above to

$$([0, a/(a + (b - a)c)], [a/(a + (b - a)c), 1]),$$

the pair obtained by taking just the first two sub-intervals of  $\mathbf{a} \circ_2 \mathbf{c}$  and rescaling to length 1,



In other words,  $p$  and  $q$  decompose the composition  $\circ_2$  into two steps,

$$\mathbf{a} \circ_2 \mathbf{c} = p(\mathbf{a}, \mathbf{c}) \circ_1 q(\mathbf{a}, \mathbf{c}).$$



Using  $p \times q: D \times D \rightarrow D \times \mathfrak{A}(2)$  and the  $\mathfrak{A}$ -algebra structure of  $R$ , we get a map  $(D_+ \wedge R) \wedge_S (D_+ \wedge R) \cong (D \times D)_+ \wedge R \wedge_S R \rightarrow D_+ \wedge \mathfrak{A}(2)_+ \wedge R \wedge_S R \rightarrow D_+ \wedge R$ . (In words, we multiply the  $D$  factors using  $p$  and the  $R$  factors according to  $q$ .) Easy computations show that this extends to a map  $A \wedge_S A \rightarrow A$  that is associative and unital (with unit  $S \rightarrow A$  induced by  $\mathbf{1} \in \bar{D}$ ), making  $A$  an associative  $S$ -algebra. The construction  $A = AR$  is clearly functorial in  $\mathfrak{A}$ -algebra maps of  $R$ , and we get the following proposition.

**Proposition 2.4.** *The construction  $A$  above defines a functor from  $\mathfrak{A}$ -algebras to associative  $S$ -algebras.*

The inclusion of  $D$  in  $\mathfrak{A}(2)$  and the inclusion of  $\bar{D}$  in  $\mathfrak{A}(1)$  induce a natural map  $\phi$  of  $S$ -modules under  $S$  from  $AR$  to  $UR$ .

**Theorem 2.5.** *The map  $\phi: AR \rightarrow UR$  is a natural isomorphism of associative  $S$ -algebras.*

*Proof.* For  $m > 0$ , let  $f_m: \mathfrak{A}(m+1) \rightarrow D \times \mathfrak{A}(m)$  denote the map that sends  $([x_1, y_1], \dots, [x_{m+1}, y_{m+1}])$  of  $\mathfrak{A}(m+1)$  to

$$([0, x_{m+1}], [x_{m+1}, y_{m+1}]), ([x_1/x_{m+1}, y_1/x_{m+1}], \dots, [x_m/x_{m+1}, y_m/x_{m+1}]),$$

and let  $f_0$  be the identity map  $\mathfrak{A}(1) = \bar{D}$ . Then for every  $m > 0$ ,  $j = j_1 + \dots + j_m > 0$ , the following diagram commutes,

$$\begin{array}{ccc} \mathfrak{A}(m+1) \times \mathfrak{A}(j_1) \times \dots \times \mathfrak{A}(j_m) & \xrightarrow{\circ} & \mathfrak{A}(j+1) \\ f_m \times \text{id} \downarrow & & \downarrow f_j \\ D \times \mathfrak{A}(m) \times \mathfrak{A}(j_1) \times \dots \times \mathfrak{A}(j_m) & \xrightarrow{\text{id} \times \circ} & D \times \mathfrak{A}(j) \end{array}$$

and the analogous diagram for  $j = 0$  commutes. It follows that the composite

$$\mathfrak{A}(m+1)_+ \wedge R^{(m)} \rightarrow D_+ \wedge \mathfrak{A}(m)_+ \wedge R^{(m)} \rightarrow D_+ \wedge R$$

for  $m > 0$  and the identity map  $\mathfrak{A}(1)_+ \wedge S = \bar{D}_+ \wedge S$  induce a map  $\epsilon: UR \rightarrow A$ , and it is easy to see from the definition of the  $S$ -algebra structures that  $\epsilon$  is a map of associative  $S$ -algebras. The composite  $\epsilon \circ \phi$  is the identity on  $A$ . Since the composite of  $f_m$  with  $\circ_1$  is the identity on  $\mathfrak{A}(m+1)$  for  $m > 0$ , the defining map

$$\bigvee_m \mathfrak{A}(m+1)_+ \wedge R^{(m)} \rightarrow UR$$

factors through  $\phi \circ \epsilon$ , and so  $\phi \circ \epsilon$  is the identity on  $UR$ .  $\square$

We close this section with the proof of Theorem 1.1: We show that the canonical map of  $UR$ -modules  $UR \rightarrow R$  induced by the inclusion of the unit  $S \rightarrow R$  is a homotopy equivalence of  $S$ -modules. In terms of the model  $A$  above, we can identify this as the composite map

$$\chi: A \rightarrow \mathfrak{A}(1)_+ \wedge R \rightarrow R.$$

induced by forgetting the *second* sub-interval in  $D$  and the  $\mathfrak{A}$ -algebra multiplication map  $\mathfrak{A}(1)_+ \wedge R \rightarrow R$ . We obtain a map back  $\psi: R \rightarrow A$  as the composite

$$R \rightarrow D_+ \wedge R \rightarrow A$$

induced by the inclusion of  $([0, 1/2], [1/2, 1])$  in  $D$ . (This map of  $S$ -modules is clearly not a map of  $UR$ -modules.) We have a homotopy  $H_t$  from  $\psi \circ \chi$  to the identity on  $A$  induced by the linear homotopy

$$H_t([0, c], [c, d]) = ([0, 1/2 + t(c - 1/2)], [1/2 + t(c - 1/2), (1 - t) + td])$$

on  $D$  (and  $\bar{D}$ ); note that  $(1 - t) + td > 1/2 + t(c - 1/2)$  since  $(1 - t)/2 + t(d - c) > 0$ . On the other side, we have a homotopy  $G_t$  from  $\chi \circ \psi$  to the identity on  $R$  induced by the path  $G_t = ([0, 1/2 + t/2])$  in  $\mathfrak{A}(1)$  and the  $\mathfrak{A}$ -algebra multiplication. This completes the proof of Theorem 1.1.

### 3. THE INTERCHANGE OPERATIONS

This section constructs the interchange operations and studies them from the perspective of the homotopy theory of  $R$ -modules. Specifically, we prove Theorems 1.5 and 1.7, which let us understand the left derived functors and their compositions. We begin with the point set construction. Throughout this section  $n$  and the  $\mathfrak{C}_n$ -algebra  $R$  remain fixed, but we note that all constructions are functorial in the  $\mathfrak{C}_n$ -algebra  $R$  and in the inclusions  $\mathfrak{C}_n \rightarrow \mathfrak{C}_{n'}$  for  $n' > n$ .

For  $\ell, m \geq 0$ , let  $\rho: \mathfrak{A}(\ell) \times \mathfrak{C}_{n-1}(m) \rightarrow \mathfrak{C}_n(\ell m)$  be the map that takes the pair

$$([a^i, b^i] \mid 1 \leq i \leq \ell), ([x_1^j, y_1^j] \times \cdots \times [x_{n-1}^j, y_{n-1}^j] \mid 1 \leq j \leq m)$$

to the sequence of sub-cubes of  $[0, 1]^n$ ,

$$[a^i, b^i] \times [x_1^j, y_1^j, 1] \times \cdots \times [x_{n-1}^j, y_{n-1}^j],$$

(for  $1 \leq i \leq \ell$ ,  $1 \leq j \leq m$ ), labelled in lexicographical order in  $(i, j)$ . As an abbreviation of this notation, write  $\rho_1: \mathfrak{C}_{n-1}(m) \rightarrow \mathfrak{C}_n(m)$  for  $\rho(\mathbf{1}, -)$ , where  $\mathbf{1}$  denotes the identity element of  $\mathfrak{A}(1)$ ,

$$\begin{aligned} \rho_1 \left( ([x_1^j, y_1^j] \times \cdots \times [x_{n-1}^j, y_{n-1}^j] \mid 1 \leq j \leq m) \right) \\ = ([0, 1] \times [x_1^j, y_1^j] \times \cdots \times [x_{n-1}^j, y_{n-1}^j] \mid 1 \leq j \leq m) \in \mathfrak{C}_n(m). \end{aligned}$$

Since for any element  $\mathbf{c}$  of  $\mathfrak{C}_{n-1}(m)$ ,  $\rho_1(\mathbf{c})$  is an element of  $\mathfrak{C}_n(m)$ , it specifies a map of  $S$ -modules  $R^{(m)} \rightarrow R$ . The key fact we need is the following.

**Proposition 3.1.** *For any  $\mathbf{c}$  in  $\mathfrak{C}_{n-1}(m)$ , the map  $R^{(m)} \rightarrow R$  induced by  $\rho_1(\mathbf{c})$  is a map of  $\mathfrak{A}$ -algebras.*

The proof consists of observing that for any  $\mathbf{a}$  in  $\mathfrak{A}(\ell)$ , both composites in the diagram

$$\begin{array}{ccc} (R^{(m)})^{(\ell)} & \xrightarrow{(\rho_1(\mathbf{c}))^{(\ell)}} & R^{(\ell)} \\ \mathbf{a} \downarrow & & \downarrow \mathbf{a} \\ R^{(m)} & \xrightarrow{\rho_1(\mathbf{c})} & R \end{array}$$

are the induced map of  $\rho(\mathbf{a}, \mathbf{c}): R^{(\ell m)} \rightarrow R$  under the isomorphism  $R^{(\ell m)} \cong (R^{(m)})^{(\ell)}$  (using the implicit lexicographical order).

Associated to the map of  $\mathfrak{A}$ -algebras  $\rho_1(\mathbf{c}): R^{(m)} \rightarrow R$ , we get a map of enveloping algebras  $U(R^{(m)}) \rightarrow UR$ . Concretely, in terms of the models  $A$  of the previous section, this is induced by the map

$$D_+ \wedge R^{(m)} \longrightarrow D_+ \wedge R$$

that performs  $\rho_1(\mathbf{c})$  on the  $R$  factors and the identity on  $D$ . More generally, for any space  $X$  and map  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$ , we get a family of maps of  $S$ -algebras  $U(R^{(m)}) \rightarrow UR$ , which by neglect of structure gives a family of  $(UR, U(R^{(m)}))$ -bimodule structures on  $UR$ , or equivalently, a  $(UR, U(R^{(m)}))$ -bimodule structure on  $UR \wedge X_+$ . Concretely, the right  $U(R^{(m)})$ -action map is induced by the map

$$(D_+ \wedge R) \wedge X_+ \wedge_S (D_+ \wedge R^{(m)}) \longrightarrow (D_+ \wedge R) \wedge_S (D_+ \wedge_S R) \longrightarrow D_+ \wedge R,$$

which is the composite of the map  $X_+ \wedge R^{(m)} \rightarrow R$  induced by  $\rho_1(f)$  and the multiplication on  $D_+ \wedge R$ .

**Notation 3.2.** For  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$ , write  $URf$  for  $UR \wedge X_+$  with the  $(UR, U(R^{(m)}))$ -bimodule structure above.

We have a canonical map of  $S$ -algebras  $U(R^{(m)}) \rightarrow (UR)^{(m)}$ , which formally is induced by the identification of  $(UR)^{(m)}$  as the left module enveloping algebra of  $R^{(m)}$  as an  $\mathfrak{A}^m$ -algebra. More concretely, it is induced by the map

$$D_+ \wedge R^{(m)} \longrightarrow (D_+ \wedge R)^{(m)},$$

which performs the diagonal map on  $D$ . We use this to regard the smash product over  $S$  of left  $UR$ -modules

$$M_1 \wedge_S \cdots \wedge_S M_m$$

as a left  $U(R^{(m)})$ -module. In this case, the left  $U(R^{(m)})$ -module structure on  $M_1 \wedge_S \cdots \wedge_S M_m$  is induced by the diagonal map on  $\mathfrak{A}$  and the left  $R$ -module structure maps on the  $M_i$ :

$$\begin{aligned} & \mathfrak{A}(j+1)_+ \wedge (R^{(m)})^{(j)} \wedge_S M_1 \wedge_S \cdots \wedge_S M_m \\ & \longrightarrow \mathfrak{A}(j+1)_+^m \wedge (R^{(m)})^{(j)} \wedge_S M_1 \wedge_S \cdots \wedge_S M_m \\ & \cong (\mathfrak{A}(j+1)_+ \wedge R^{(j)} \wedge_S M_1) \wedge_S \cdots \wedge_S (\mathfrak{A}(j+1)_+ \wedge R^{(j)} \wedge_S M_m) \\ & \longrightarrow M_1 \wedge_S \cdots \wedge_S M_m. \end{aligned}$$

**Construction 3.3.** For  $f: X \rightarrow \mathfrak{C}_{n-1}$ , let  $\Lambda_f$  be the functor  $(\mathcal{M}_R)^m \rightarrow \mathcal{M}_R$  defined by

$$\Lambda_f(M_1, \dots, M_m) = URf \wedge_{U(R^{(m)})} (M_1 \wedge_S \cdots \wedge_S M_m).$$

Having constructed the point-set operations, we now study them from the perspective of the homotopy theory of  $R$ -modules. Following Convention 2.3, we understand homotopical concepts in  $R$ -modules in terms of  $UR$ -modules. Here we begin to take advantage of the technical properties of EKMM  $S$ -modules: Because weak equivalences between cofibrant  $R$ -modules are homotopy equivalences, and because topologically enriched functors preserve homotopies, left derived functors of topologically enriched functors always exist and are formed by applying the point-set functor to a cofibrant approximation. Equivalently, and more conveniently for us, we can work in terms of  $R$ -modules that are homotopy equivalent to cofibrant  $R$ -modules. We use the following terminology.

**Definition 3.4.** A *homotopy cofibrant*  $R$ -module is an  $R$ -module that is homotopy equivalent to a cofibrant  $R$ -module, or equivalently [6, VII.4.15], homotopy equivalent to a cell  $R$ -module [6, III§2].

The  $E_n$  interchange operations  $\Lambda_f$  are topologically enriched, and in fact are enriched over  $S$ -modules as functors of several variables. Thus, their left derived functors exist and are calculated by homotopy cofibrant approximation. In fact, the  $S$ -module enriched left derived functors [13, §5] exist.

**Proposition 3.5.** *For any  $f: X \rightarrow \mathfrak{C}_{n-1}(m)$ , the left derived functor of  $\Lambda_f$  exists, is computed by approximating by a weakly equivalent homotopy cofibrant object, and is enriched over the stable category.*

The main tool we have to study the homotopy theory of the  $E_n$  interchange operations is the following lemma proved at the end of the section.

**Lemma 3.6.** *The canonical map  $U(R^{(m)}) \rightarrow (UR)^{(m)}$  is a homotopy equivalence of left  $U(R^{(m)})$ -modules.*

For a cell  $(UR)^{(m)}$ -module  $M$ , applying the previous lemma inductively, we see that  $M$  is homotopy cofibrant as a  $U(R^{(m)})$ -module. This implies the following proposition.

**Proposition 3.7.** *Let  $M$  be a  $(UR)^{(m)}$ -module. If  $M$  is homotopy cofibrant as a  $(UR)^{(m)}$ -module, then it is homotopy cofibrant as a  $U(R^{(m)})$ -module.*

We can now prove Theorems 1.5 and 1.7.

*Proof of Theorem 1.5.* Write  $M$  for the left  $(UR)^{(j)}$ -module  $M_1 \wedge_S \cdots \wedge_S M_j$  in the statement. The map in question is induced by applying  $(-) \wedge_{(UR)^{(j)}} M$  to the map of  $(UR, (UR)^{(j)})$ -bimodules

$$\begin{aligned} URf \wedge_{U(R^{(m)})} (URg_1 \wedge_S \cdots \wedge_S URg_m) \wedge_{U(R^{(j_1)}) \wedge_S \cdots \wedge_S U(R^{(j_m)})} (UR)^{(j)} \\ \longrightarrow UR(f \circ (g_1, \dots, g_m)) \wedge_{U(R^{(j)})} (UR)^{(j)}. \end{aligned}$$

Since by hypothesis,  $M$  is a cofibrant left  $(UR)^{(j)}$ -module, it suffices to show that the map above is a weak equivalence. By the Lemma 3.6, it suffices to show that the map

$$URf \wedge_{U(R^{(m)})} (URg_1 \wedge_S \cdots \wedge_S URg) \longrightarrow UR(f \circ (g_1, \dots, g_m))$$

is a weak equivalence. Since this is the map

$$UR \wedge X_+ \wedge_{U(R^{(m)})} (UR)^{(m)} \wedge (Y_1 \times \cdots \times Y_m)_+ \longrightarrow UR \wedge (X \times Y_1 \times \cdots \times Y_m)_+,$$

we see that it is a weak equivalence by applying Lemma 3.6 a second time.  $\square$

*Proof of Theorem 1.7.* Applying Proposition 3.7, we can choose a cell  $U(R^{(m)})$ -module homotopy equivalent to  $M_1 \wedge_S \cdots \wedge_S M_m$ , and then it suffices to show that  $URf \wedge_{U(R^{(m)})} M$  is homotopy cofibrant. Working inductively with the cell structure, it suffices to check the case when  $M$  is a single cell  $U(R^{(m)}) \wedge_S S_S^n$  (in the notation of [6, II.1.7]). In this case,

$$URf \wedge_{U(R^{(m)})} (U(R^{(m)}) \wedge_S S_S^n) \cong URf \wedge S_S^n = (UR \wedge X_+) \wedge_S S_S^n \cong UR \wedge_S S_S^n \wedge X_+$$

is homotopy cofibrant.  $\square$

We close this section with the proof of Lemma 3.6. The proof requires the construction of  $U(R^{(m)})$  as  $A(R^{(m)})$  in Section 2. We begin by describing maps and homotopies on  $D$ .

Write  $\Delta$  for the diagonal map  $D \rightarrow D^m$  and consider the map  $g: D^m \rightarrow D$  defined by

$$g: (\mathbf{c}_1, \dots, \mathbf{c}_m) = ([0, c_1], [c_1, d_1]), \dots, ([0, c_m], [c_m, d_m]) \mapsto ([0, c], [c, d])$$

where  $c = \max\{c_1, \dots, c_m\}$  and  $d - c = \min\{d_1 - c_1, \dots, d_m - c_m\}$ . We have that  $g \circ \Delta$  is the identity on  $D$ . We obtain a homotopy  $h_t$  on  $D^m$  from  $\Delta \circ g$  to the identity defined by the linear homotopy in each coordinate

$$h_t(\mathbf{c}_1, \dots, \mathbf{c}_m)_i = ([0, c + t(c_i - c)], [c + t(c_i - c), d + t(d_i - d)]).$$

Note that  $d + t(d_i - d) > c + t(c_i - c)$  since  $(1 - t)(d - c) + t(d_i - c_i) > 0$ . Analogous formulas define maps and homotopies when one or more factors of  $D$  are replaced by  $\bar{D}$  (with the analogue of  $g$  or a coordinate of the analogue of  $\Delta \circ g$  landing in  $\bar{D}$  when appropriate).

Next we see how the homotopies interact with the maps  $p$  and  $q$  in the construction of  $A$ . For  $\mathbf{a} = ([0, a], [a, b])$  in  $D$ , we have

$$p(\mathbf{a}, h_t(\mathbf{c}_1, \dots, \mathbf{c}_m)_i) = ([0, x], [x, y]),$$

where

$$\begin{aligned} x &= a + (b - a)(c + t(c_i - c)) = a + (b - a)c + t(b - a)(c_i - c) \\ &= a + (b - a)c + t((a + (b - a)c_i) - (a + (b - a)c)) \end{aligned}$$

and

$$\begin{aligned} y &= a + (b - a)(d + t(d_i - d)) = a + (b - a)d + t(b - a)(d_i - d) \\ &= a + (b - a)d + t((a + (b - a)d_i) - (a + (b - a)d)). \end{aligned}$$

Since  $\max\{a + (b - a)c_i\}$  is  $a + (b - a)c$  and  $\min\{(a + (b - a)d_i) - (a + (b - a)c_i)\}$  is  $(b - a)(d - c)$ , we see that

$$p(\mathbf{a}, h_t(\mathbf{c}_1, \dots, \mathbf{c}_m)_i) = h_t(p(\mathbf{a}, \mathbf{c}_1), \dots, p(\mathbf{a}, \mathbf{c}_m))_i.$$

Likewise, since

$$\frac{a}{(a + (b - a)(c + t(c_i - c)))} = \frac{a}{(a + (b - a)c + t((a + (b - a)c_i) - (a + (b - a)c)))},$$

we have

$$q(\mathbf{a}, h_t(\mathbf{c}_1, \dots, \mathbf{c}_m)_i) = h_t(q(\mathbf{a}, \mathbf{c}_1), \dots, q(\mathbf{a}, \mathbf{c}_m))_i.$$

Putting this together with  $R^{(m)}$ , we get a map

$$g: D_+^m \wedge R^{(m)} \longrightarrow D_+ \wedge R^{(m)}$$

and a homotopy

$$h_t: D_+^m \wedge R^{(m)} \longrightarrow D_+^m \wedge R^{(m)}.$$

The formulas above imply that these are compatible with the left action of  $D_+ \wedge R^{(m)}$ . Moreover, these are compatible with the analogous maps obtained by replacing one or more factors of  $D$  by  $\bar{D}$  and the corresponding factor of  $R$  with  $S$ . Passing to iterated pushouts, we get a map

$$g: (UR)^{(m)} \longrightarrow U(R^{(m)})$$

and a homotopy

$$h_t: (UR)^{(m)} \longrightarrow (UR)^{(m)}$$

compatible with the left  $U(R^{(m)})$ -action.

## 4. PROOF OF THE MAIN THEOREM

In this section, we prove the Main Theorem, which amounts to specifying constructions and verifying coherence diagrams. For the case of an  $E_2$  algebra, we construct the smash product in 4.1, the right adjoint function modules in 4.2, and compare the categories of left and right modules in 4.3. We construct the unit and associativity isomorphisms in 4.4 and 4.5, and prove the unit and associativity coherence in 4.6. For the  $E_3$  case, we construct the braid isomorphism and prove its coherence in 4.7, and for the  $E_4$  case, we show that the braid isomorphism is a symmetry isomorphism in 4.8.

**4.1. The Smash Product.** Let  $\mu$  be the element  $([0, 1/2], [1/2, 1])$  in  $\mathfrak{C}_1(2)$ . For left  $R$ -modules  $M, N$ , define

$$M \wedge_R N = \Lambda_\mu(M, N).$$

Let  $\wedge_R: \mathcal{D}_R \times \mathcal{D}_R \rightarrow \mathcal{D}_R$  be the left derived functor.

**4.2. The Function Modules.** For a left  $R$ -module  $M$ , let

$$M^\ell = UR\mu \wedge_{U(R^{(2)})} (UR \wedge_S M) \quad \text{and} \quad M^r = UR\mu \wedge_{U(R^{(2)})} (M \wedge_S UR)$$

in the notation of 3.2. These are  $(UR, UR)$ -bimodules using the left  $UR$ -module structure on  $UR\mu$  and the right  $UR$ -module structure on  $UR$ . Clearly, the functors

$$F_{UR}(M^\ell, -) \quad \text{and} \quad F_{UR}(M^r, -)$$

are right adjoint to the point-set functors  $(-) \wedge_R M$  and  $M \wedge_R (-)$  defined above. Now assume  $M$  is homotopy cofibrant. Then  $M^\ell$  and  $M^r$  are homotopy cofibrant as left  $UR$ -modules, and so these functors preserve weak equivalences between arbitrary  $UR$ -modules; therefore, their right derived functors exist. Since for any homotopy cofibrant  $N$ ,  $N \wedge_R M$  and  $M \wedge_R N$  are homotopy cofibrant, an easy check shows that the right derived functors of  $F_{UR}(M^\ell, -)$  and  $F_{UR}(M^r, -)$  remain adjoint to the left derived functors of  $(-) \wedge_R M$  and  $M \wedge_R (-)$ .

**4.3. Comparison of Left and Right Modules.** Forgetting the left  $UR$ -module structure on  $M^r$  defines a functor  $r$  from  $\mathcal{M}_{UR}$  to  $\mathcal{M}_{UR^{\text{op}}}$ , and a derived functor from  $\mathcal{D}_R$  to  $\mathcal{D}_{R^{\text{op}}}$ . By construction, (the underlying  $S$ -module of) the smash product above is the composite of  $r$  with the balanced product of a left and right  $UR$ -module

$$\Lambda_\mu(M, N) = rM \wedge_{UR} N.$$

To see that  $r$  induces an equivalence on derived categories, we can rewrite  $r$  as

$$rM = (UR\mu \wedge_{U(R^{(2)})} (UR \wedge_S UR)) \wedge_{UR} M.$$

Writing  $W$  for  $UR\mu \wedge_{U(R^{(2)})} (UR \wedge_S UR)$ , we can identify the derived functor as

$$r(-) = \text{Tor}_{UR}(UR^{\text{op}}W, -)$$

in the notation of [13]. Applying [13, 8.5], we see that the right adjoint  $\text{Ext}_{UR^{\text{op}}}(W_{UR}, -)$  exists. Since  $W$  is weakly equivalent to  $UR$  in each of its right  $UR$ -module structures, both derived functors are naturally isomorphic to the identity on the underlying  $S$ -modules. In particular, it follows that the unit and counit of the derived adjunction are isomorphisms and these functors are inverse equivalences.

**4.4. The Unit Isomorphisms.** Since  $UR \rightarrow R$  is a weak equivalence, we have  $UR \wedge_S S_S \rightarrow R$  as a cofibrant approximation. Let

$$\mu_0^\ell = ([1/2, 1]) \quad \text{and} \quad \mu_0^r = ([0, 1/2]),$$

elements of  $\mathfrak{C}_1(1)$ . The maps of  $\mathfrak{A}$ -algebras

$$i_1: R = S \wedge_S R \longrightarrow R \wedge_S R \quad \text{and} \quad i_2: R = R \wedge_S S \longrightarrow R \wedge_S R$$

allow us to regard  $UR\mu$  as a  $(UR, UR)$ -bimodule two different ways, and we have canonical isomorphisms of left  $UR$ -modules

$$UR\mu \wedge_{UR, i_1} M \cong UR\mu_0^\ell \wedge_{UR} M \quad \text{and} \quad UR\mu \wedge_{UR, i_2} M \cong UR\mu_0^r \wedge_{UR} M.$$

Letting  $\eta^\ell$  and  $\eta^r$  denote the linear paths in  $\mathfrak{C}_1$  from  $\mu_0^\ell$  and  $\mu_0^r$  to  $\mathbf{1} = ([0, 1])$ , we then have natural maps

$$\begin{aligned} \Lambda_\mu(UR \wedge S_S, M) &\longleftarrow \Lambda_{\mu_0^\ell}(M) \wedge_S S_S \longrightarrow \Lambda_{\eta^\ell}(M) \longleftarrow \Lambda_{\{\mathbf{1}\}}(M) = M \\ \Lambda_\mu(M, UR \wedge S_S) &\longleftarrow \Lambda_{\mu_0^r}(M) \wedge_S S_S \longrightarrow \Lambda_{\eta^r}(M) \longleftarrow \Lambda_{\{\mathbf{1}\}}(M) = M, \end{aligned}$$

in which all maps are weak equivalences when  $M$  is homotopy cofibrant. These are the left and right unit isomorphisms  $\lambda$  and  $\rho$ .

**4.5. The Associativity Isomorphism.** As indicated in Section 1, for an appropriate path  $\alpha$  in  $\mathfrak{C}_1(3)$ , the associativity isomorphism is the zigzag (1.6),

$$\begin{array}{ccc} \Lambda_{\mu \circ_1 \mu}(L, M, N) & \longrightarrow & \Lambda_\alpha(L, M, N) \longleftarrow \Lambda_{\mu \circ_2 \mu}(L, M, N) \\ \downarrow & & \downarrow \\ \Lambda_\mu(\Lambda_\mu(L, M), N) & & \Lambda_\mu(L, \Lambda_\mu(M, N)), \end{array}$$

in which all the maps are weak equivalences when  $L$ ,  $M$ , and  $N$  are homotopy cofibrant.

**4.6. The Coherence Diagrams.** For coherence of associativity, we want to show that the pentagon diagram in  $\mathscr{D}_R$

$$\begin{array}{ccc} & (K \wedge_R L) \wedge_R (M \wedge_R N) & \\ \nearrow & & \searrow \\ ((K \wedge_R L) \wedge_R M) \wedge_R N & & K \wedge_R (L \wedge_R (M \wedge_R N)) \\ \downarrow & & \uparrow \\ (K \wedge_R (L \wedge_R M)) \wedge_R N & \longrightarrow & K \wedge_R ((L \wedge_R M) \wedge_R N) \end{array}$$

commutes. The paths

$$\alpha \circ_1 \mu, \quad \mu \circ_1 \alpha, \quad \alpha \circ_2 \mu, \quad \mu \circ_2 \alpha, \quad \alpha \circ_3 \mu$$

specify a map from the boundary of the pentagon into  $\mathfrak{C}_1(4)$ , which can be filled in to a map  $\pi$  from the pentagon into  $\mathfrak{C}_1(4)$  by the contractibility of the components of  $\mathfrak{C}_1(4)$  (or by making explicit choices). We then have the following commutative

diagram of weak equivalences of  $(UR, U(R^4))$ -bimodules.

$$\begin{array}{ccccc}
 & & UR(\mu \circ (\mu, \mu)) & & \\
 & \swarrow & \downarrow & \searrow & \\
 UR(\alpha \circ_1 \mu) & & & & UR(\alpha \circ_3 \mu) \\
 \uparrow & \searrow & & \swarrow & \uparrow \\
 UR(\mu \circ_1 (\mu \circ_1 \mu)) & \longrightarrow & UR\pi & \longleftarrow & UR(\mu \circ_2 (\mu \circ_2 \mu)) \\
 \downarrow & \nearrow & \uparrow & \nwarrow & \downarrow \\
 UR(\mu \circ_1 \alpha) & & & & UR(\mu \circ_2 \alpha) \\
 \uparrow & \nearrow & \uparrow & \nwarrow & \uparrow \\
 UR(\mu \circ_1 (\mu \circ_2 \mu)) & \longrightarrow & UR(\alpha \circ_2 \mu) & \longleftarrow & UR(\mu \circ_2 (\mu \circ_1 \mu))
 \end{array}$$

We see that both composites of derived functors

$$\Lambda_\mu \circ_1 (\Lambda_\mu \circ_1 \Lambda_\mu) \longrightarrow \Lambda_\mu \circ_2 (\Lambda_\mu \circ_2 \Lambda_\mu)$$

in  $\mathcal{D}_R$  are represented by the zigzag

$$\Lambda_\mu \circ_1 (\Lambda_\mu \circ_1 \Lambda_\mu) \longleftarrow \Lambda_{\mu \circ_1 (\mu \circ_1 \mu)} \longrightarrow \Lambda_\pi \longleftarrow \Lambda_{\mu \circ_2 (\mu \circ_2 \mu)} \longrightarrow \Lambda_\mu \circ_2 (\Lambda_\mu \circ_2 \Lambda_\mu),$$

and so coincide. It follows that the associativity coherence pentagon in  $\mathcal{D}_R$  commutes.

For the coherence of the unit, we want to show that the triangle diagram in  $\mathcal{D}_R$

$$\begin{array}{ccc}
 M \wedge_R (R \wedge_R N) & \xrightarrow{\alpha} & (M \wedge_R R) \wedge_R N \\
 \searrow \text{id} \wedge \lambda & & \swarrow \rho \wedge \text{id} \\
 & M \wedge_R N &
 \end{array}$$

commutes. Letting  $\mathbf{i}$  denote the unique element of  $\mathfrak{C}_1(0)$ , the paths

$$\mu \circ_2 \eta^\ell, \quad \mu \circ_1 \eta^r, \quad \alpha \circ (\mathbf{1}, \mathbf{i}, \mathbf{1})$$

specify a map from the boundary of the triangle to  $\mathfrak{C}_1(2)$  that fills in by contractibility. An argument like the previous one then shows that the unit triangle in  $\mathcal{D}_R$  commutes.

**4.7. The Braid Isomorphism.** We now assume that  $R$  is a  $\mathfrak{C}_3$ -algebra. Since  $\mathfrak{C}_2(2)$  is connected, we can choose a path  $\sigma$  from

$$\mu = ([0, 1/2] \times [0, 1], [1/2, 1] \times [0, 1])$$

to

$$\mu\tau = ([1/2, 1] \times [0, 1], [0, 1/2] \times [0, 1]).$$

(Such a path is illustrated in Section 1.) We then get a braid isomorphism in  $\mathcal{D}_R$  from the zigzag

$$\Lambda_\mu(M, N) \longrightarrow \Lambda_\sigma(M, N) \longleftarrow \Lambda_{\mu\tau}(M, N)$$

and the isomorphism  $\Lambda_{\mu\tau}(M, N) \cong \Lambda_\mu(N, M)$  induced (upon passing to coequalizers) by the isomorphism

$$UR \wedge_S M \wedge_S N \longrightarrow UR \wedge_S N \wedge_S M.$$



We need to show that hexagon diagram in  $\mathcal{D}_R$

$$\begin{array}{ccc}
 & (M \wedge_R L) \wedge_R N \xrightarrow{\alpha} M \wedge_R (L \wedge_R N) & \\
 \sigma \wedge \text{id} \nearrow & & \searrow \text{id} \wedge \sigma \\
 (L \wedge_R M) \wedge_R N & & M \wedge_R (N \wedge_R L) \\
 \alpha \searrow & & \nearrow \alpha \\
 & L \wedge_R (M \wedge_R N) \xrightarrow{\sigma} (M \wedge_R N) \wedge_R L &
 \end{array}$$

and the analogous hexagon with  $\sigma$  replaced with its inverse (or equivalently,  $\alpha$  replaced by its inverse) commute. The paths

$$\mu \circ_1 \sigma, \quad \alpha, \quad \mu \circ_2 \sigma, \quad \alpha, \quad \sigma \circ_1 \mu, \quad \alpha$$

join together to define a map from the boundary of a hexagon into  $\mathfrak{C}_2(3)$ . The fundamental group of  $\mathfrak{C}_2(3)$  is the braid group  $B_3$  on 3 strands, and the braid relation on  $\pi_1(\mathfrak{C}_2(3))$  implies that this can be filled in to a map from the hexagon. The remainder of the proof follows just as in the arguments in the previous subsection, and the other case is similar.

**4.8. The Symmetry Isomorphism.** We now assume that  $R$  is a  $\mathfrak{C}_4$ -algebra. We want to show that  $\sigma_{M,N} = \sigma_{N,M}^{-1}$  in  $\mathcal{D}_R$ , that is, that the composite map in  $\mathcal{D}_R$

$$M \wedge_R N \xrightarrow{\sigma} N \wedge_R M \xrightarrow{\sigma} M \wedge_R N$$

is the identity. This follows from the fact that the loop obtained from the paths

$$\sigma, \quad \sigma\tau$$

in  $\mathfrak{C}_2(2)$  is contractible in  $\mathfrak{C}_3(2)$ . (This loop is not contractible in  $\mathfrak{C}_2(2)$ , but generates  $\pi_1(\mathfrak{C}_2(2)) = B_2 \cong \mathbb{Z}$ .)

## 5. THE LAX MONOIDAL SMASH PRODUCT

Previous sections have concentrated on the monoidal structure on the derived category of left  $R$ -modules for a  $\mathfrak{C}_2$ -algebra  $R$ . In this section, we study the structure that arises on the point-set category of  $R$ -modules. Although we do not get a true monoidal structure, we do get some kind of weaker structure. The purpose of this section is to describe this structure and to outline an approach to constructing it.

We organize the discussion in terms of the *lax monoidal structures* of [11, 3.1.1]. Such a structure on a category  $\mathcal{M}$  consists of functors

$$\bigotimes_n: \mathcal{M}^n \longrightarrow \mathcal{M}$$

for  $n$  a natural number (including zero) and natural transformations  $\gamma, \iota$  satisfying certain coherence conditions. These coherence conditions are most concisely specified in terms of trees and edge contractions: An arbitrary composite of the functors  $\bigotimes$  can be viewed as a planar tree with leaves labelled either by an object of  $\mathcal{M}$  or by  $\bigotimes_0$ . The natural transformations  $\gamma$  compose

$$\begin{aligned}
 \bigotimes_m(M_1, \dots, M_{i-1}, \bigotimes_n(N_1, \dots, N_n), M_{i+1}, \dots, M_m) \\
 \longrightarrow \bigotimes_{m+n-1}(M_1, \dots, M_{i-1}, N_1, \dots, N_n, M_{i+1}, \dots, M_m),
 \end{aligned}$$

and so contract an internal edge or edge ending in a leaf labelled  $\otimes_0$ ; the natural transformation  $\iota$  is a map

$$M \longrightarrow \otimes_1 M$$

and so is essentially an edge insertion, converting a node into two nodes with an edge connecting them. The coherence conditions are that (1) all sequences of edge contractions that take a given planar labelled tree to another given planar labelled tree performs the same natural transformation, and (2) an edge insertion followed by an edge contraction of the inserted edge is the identity. See [11, 3.1.1] for a formulation not involving tree operations.

A fundamental property of the theory of lax monoidal categories is Theorem 3.1.6 of [11]: A lax monoidal category in which the natural transformations  $\gamma$  and  $\iota$  are isomorphisms is equivalent to a (strict) monoidal category. From the perspective of homotopy theory, we can view a lax monoidal category where the natural transformations  $\gamma$  and  $\iota$  are weak equivalences (say, after restricting to homotopy cofibrant objects) as an up-to-coherent-homotopy version of a monoidal category.

In the context of an  $E_2$  algebra  $R$ , to give the idea of how to construct the lax monoidal structure on  $\mathcal{M}_R$ , we begin with a non-unital version, omitting  $\otimes_0$ . For this version of the construction, we can also take  $\otimes_1$  and  $\iota$  to be the identity functor and map. Then we only need to treat  $\otimes_n$  for  $n \geq 2$ . The composite operations are then the planar trees labelled by objects of  $\mathcal{M}_R$  where all internal nodes have valence 2 or more. Choosing a map of operads  $\phi$  from the Stasheff operad  $\mathfrak{K}$  to  $\mathfrak{A}$ , we construct  $\otimes_n$  inductively as follows. Writing  $\phi_n$  for the map  $\mathfrak{K}(n) \rightarrow \mathfrak{A}(n)$ ,  $\phi_2$  is the inclusion of a point in  $\mathfrak{A}(2)$ , and we let  $\otimes_2(M_1, M_2) = \Lambda_{\phi_2}(M_1, M_2)$ . We have that  $\mathfrak{K}(2)$  is an interval and  $\phi_3$  is a path between  $\phi_2 \circ_1 \phi_2$  and  $\phi_2 \circ_2 \phi_2$ ; we define  $\otimes_3$  as the colimit of the diagram

$$\begin{array}{ccc} \Lambda_{\phi_2 \circ_1 \phi_2}(M_1, M_2, M_3) & \longrightarrow & \Lambda_{\phi_3}(M_1, M_2, M_3) \longleftarrow \Lambda_{\phi_2 \circ_2 \phi_2}(M_1, M_2, M_3) \\ \downarrow & & \downarrow \\ \otimes_2(\otimes_2(M_1, M_2), M_3) & & \otimes_2(M_1, \otimes_2(M_2, M_3)), \end{array}$$

which we can identify as the pushout of the maps

$$\begin{aligned} \Lambda_{\phi_2 \circ_1 \phi_2}(M_1, M_2, M_3) &\longrightarrow \Lambda_{\phi_2}(\Lambda_{\phi_2}(M_1, M_2), M_3), \\ \Lambda_{\phi_2 \circ_2 \phi_2}(M_1, M_2, M_3) &\longrightarrow \Lambda_{\phi_2}(M_1, \Lambda_{\phi_2}(M_2, M_3)) \end{aligned}$$

over the Hurewicz cofibration

$$\Lambda_{\phi_2 \circ_1 \phi_2}(M_1, M_2, M_3) \vee \Lambda_{\phi_2 \circ_2 \phi_2}(M_1, M_2, M_3) \longrightarrow \Lambda_{\phi_3}(M_1, M_2, M_3).$$

In general, the polytope  $\mathfrak{K}(n)$  has a sub-face for each planar tree with all internal nodes of valence 2 or more. Thus, the boundary consists of all formal compositions (of total valence  $n$ ) of all lower valence polytopes. We can glue together the corresponding compositions of the operations  $\otimes_m$  to form an operation “ $\otimes_{\partial \mathfrak{K}(n)}$ ”. Writing  $\partial \phi_n$  for the restriction of  $\phi_n$  to the boundary of  $\mathfrak{K}(n)$ , we have a natural transformation

$$\Lambda_{\partial \phi_n}(M_1, \dots, M_n) \longrightarrow \otimes_{\partial \mathfrak{K}(n)}(M_1, \dots, M_n),$$

since we can identify  $\Lambda_{\partial \phi}$  as the colimit obtained by gluing the corresponding operations  $\Lambda_f$  obtained by composing in the operad. Moreover, both colimits are formed by corresponding iterated pushouts along Hurewicz cofibrations; by induction, when the modules  $M_1, \dots, M_n$  are homotopy cofibrant, the comparison map

on each formal composition in the boundary is a weak equivalence, and so the natural transformation above on their colimits is a weak equivalence. We define  $\bigotimes_n(M_1, \dots, M_n)$  by the pushout diagram

$$\begin{array}{ccc} \Lambda_{\partial\phi_n}(M_1, \dots, M_n) & \longrightarrow & \Lambda_{\phi_n}(M_1, \dots, M_n) \\ \downarrow & & \downarrow \\ \bigotimes_{\partial\mathfrak{K}(n)}(M_1, \dots, M_n) & \longrightarrow & \bigotimes_n(M_1, \dots, M_n). \end{array}$$

The compositions  $\gamma$  are then Hurewicz cofibrations, and are weak equivalences when the modules  $M_1, \dots, M_n$  are homotopy cofibrant.

The previous construction used the interpretation of the cells of the Stasheff operad in terms of trees, or equivalently, the fact that the Stasheff operad is the cofibrant operad on one cell in each valence (or “arity”)  $n \geq 2$ . To put the unit in, we need to use a cofibrant  $A_\infty$  operad  $\mathfrak{K}^u$  having  $\mathfrak{K}^u(0)$  contractible instead of empty. Using generating cells in valence zero reflecting the structure of the unit maps in Section 4.4, the construction above then generalizes to produce a lax monoidal structure with  $\mathfrak{K}^u(n)$  (rather than  $\mathfrak{K}(n)$ ) parametrizing the construction of  $\bigotimes_n$ . We omit the remaining details.

At the cost of weakening the point-set structure further, we get a structure even closer to the structure of  $E_n$  interchange operations from Section 3. We introduce the following “partial” version of a lax monoidal category. Again, this is easiest to explain in terms of planar trees. We write  $\mathcal{T}_n$  for the partially ordered set of planar trees with  $n$  distinguished leaves (terminal nodes), where we have a map  $T \rightarrow T'$  in  $\mathcal{T}_n$  when  $T'$  can be obtained from  $T$  by contracting internal edges (edges that end in an internal node) and/or edges ending in undistinguished leaves. We understand  $\mathcal{T}_0$  as the category with a single object (the empty tree) and morphism (the identity). We have functors

$$\circ_i: \mathcal{T}_m \times \mathcal{T}_n \longrightarrow \mathcal{T}_{m+n-1}$$

that send  $(T, T')$  to the tree that grafts  $T'$  onto  $T$  replacing the  $i$ -th distinguished leaf (counting from left to right) if  $T'$  is not the empty tree, or makes the  $i$ -th distinguished leaf undistinguished if  $T'$  is the empty tree.

**Definition 5.1.** A *partial lax monoidal structure* on a category  $\mathcal{M}$  consists of functors

$$\bigotimes_{(-)}: \mathcal{T}_n \times \mathcal{M}^n \longrightarrow \mathcal{M},$$

natural transformations

$$\eta: \bigotimes_{T \circ_i T'} \longrightarrow \bigotimes_T \circ_i \bigotimes_{T'}$$

and a natural transformation

$$\iota: \text{Id} \longrightarrow \bigotimes_{S_1}$$

(where  $S_1$  is the star with one leaf), such that the transitivity diagrams

$$\begin{array}{ccc}
 \bigotimes_{(T_1 \circ_i T_2) \circ_j T_3} & \xrightarrow{=} & \bigotimes_{T_1 \circ_{i'} (T_2 \circ_{j'} T_3)} \\
 \eta \downarrow & & \downarrow \eta \\
 \bigotimes_{T_1 \circ_i T_2} \circ_j \bigotimes_{T_3} & & \bigotimes_{T_1} \circ_{i'} \bigotimes_{T_2 \circ_{j'} T_3} \\
 \eta \downarrow & & \downarrow \eta \\
 (\bigotimes_{T_1} \circ_i \bigotimes_{T_2}) \circ_j \bigotimes_{T_3} & \xrightarrow{=} & \bigotimes_{T_1} \circ_{i'} (\bigotimes_{T_2} \circ_{j'} \bigotimes_{T_3})
 \end{array}$$

commute for all  $i, j$  (and appropriate  $i', j'$ ), and the unit diagrams

$$\begin{array}{ccc}
 \bigotimes_{S_1 \circ T} & & \bigotimes_{T \circ_i S_1} \\
 \eta \swarrow \downarrow & & \swarrow \eta \\
 \bigotimes_T & \xrightarrow{\iota} & \bigotimes_{S_1} \circ \bigotimes_T \\
 & & \downarrow \iota \\
 & & \bigotimes_T \circ_i \bigotimes_{S_1}
 \end{array}$$

commute for all  $i$ .

A partial lax monoidal category for which the natural transformations  $\eta$  are isomorphisms is equivalent to a lax monoidal category: We take  $\bigotimes_n$  to be  $\bigotimes_{S_n}$  for the stars  $S_n$ . The natural transformations  $\gamma$  are the composites

$$\bigotimes_m \circ_i \bigotimes_n \xrightarrow{\eta^{-1}} \bigotimes_{S_m \circ_i S_n} \longrightarrow \bigotimes_{m+n-1},$$

where the unlabelled arrow is the map induced by the edge contraction  $S_m \circ_i S_n \rightarrow S_{m+n-1}$ . A partial lax monoidal category for which all the structure maps (the maps  $\eta$ ,  $\iota$ , and the maps induced by maps in  $\mathcal{T}_n$ ) are weak equivalences is then another kind of up-to-coherent-homotopy version of a monoidal category.

In our context, we have the following result on the partial lax monoidal structure on the category of left modules over an  $E_2$  algebra  $R$ . The unit of this structure will be the left module  $UR$ , which is not cofibrant, but does have the property that  $UR \wedge_S S_S$  is cofibrant. In the following theorem, we say that an  $R$ -module is *nearly homotopy cofibrant* if  $(-) \wedge_S S_S$  makes it into a homotopy cofibrant  $R$ -module.

**Theorem 5.2.** *For a  $\mathfrak{C}_2$ -algebra  $R$ ,  $\mathcal{M}_R$  is a partial lax monoidal category. This structure restricts to a partial lax monoidal structure on the full subcategory of nearly homotopy cofibrant  $R$ -modules; moreover, on this subcategory, the structure maps are weak equivalences.*

Given a tree  $T$ , write

$$\mathfrak{A}(T) = \mathfrak{A}(n_1) \times \cdots \times \mathfrak{A}(n_r)$$

where  $n_1, \dots, n_r$  are the valences of the internal nodes. This, together with the operadic multiplication, makes  $\mathfrak{A}$  into functors on the categories  $\mathcal{T}_n$ , as in [8, §1]. We define

$$\bigotimes_T = \Lambda_{\mathfrak{A}(T)},$$

we take the maps  $\eta$  to be the natural transformations as constructed in (1.4), and we take  $\iota$  to be the inclusion  $\text{Id} = \Lambda_{\{1\}} \rightarrow \Lambda_{\mathfrak{A}(1)} = \bigotimes_{S_1}$ . The commutativity of the diagrams is an easy check of the definitions. As the constructions preserve homotopy

equivalences and commute with  $(-) \wedge_S S_S$ , the structure restricts to a structure on the full subcategory of nearly homotopy cofibrant modules by Theorem 1.7. The weak equivalence assertion follows from Theorem 1.5 and its proof.

**Generalizations.** For an  $E_n$  algebra  $R$ ,  $n > 2$ , one expects a point-set structure on the category of  $R$ -modules reflecting the additional structure on  $R$ . Using ideas of [5] and [1], one possibility would be some kind of lax (or partial lax) iterated monoidal category structure. Alternatively, one can view a lax monoidal structure on a category  $\mathcal{C}$  as a pseudo-functorial map of operads of categories

$$\Sigma \longrightarrow \text{End}(\mathcal{C}),$$

where  $\Sigma$  denotes the associative algebra operad  $\Sigma(n) = \Sigma_n$  (viewed as a discrete category), and  $\text{End}(\mathcal{C})$  denotes the endomorphism operad

$$\text{End}(\mathcal{C})(n) = \text{Fun}(\mathcal{C}^n, \mathcal{C})$$

of functors  $\mathcal{C}^n \rightarrow \mathcal{C}$  and natural transformations. Recent development of the theory of quasi-categories give an interpretation of  $\text{End}(\mathcal{C})$  as an operad in  $(\infty, 1)$ -categories (using a simplicial localization, singular complex, or homotopy coherent nerve construction); another formulation of the expected structure would be a map

$$\mathfrak{C}_{n-1} \longrightarrow \text{End}(\mathcal{M}_R)$$

in an appropriate homotopy category of operads of  $(\infty, 1)$ -categories. We intend these remarks as suggestive rather than rigorous and offer no further details.

**Converse Conjectures.** In the context of stable homotopy categories, under suitable technical hypotheses, the thick subcategory generated by a particular object  $X$  is equivalent to the thick subcategory of small objects in the derived category of the endomorphism ring spectrum  $\text{End}(X)$  of  $X$ . The converse conjecture in the introduction is based on the familiar principle that structure on the derived category should reflect and be reflected by structure on the ring.

In the case of an  $E_2$  ring spectrum  $R$ , we have seen above that the derived category obtains a monoidal structure and the point-set category obtains a weakened version of a monoidal structure with the unit weakly equivalent to  $R$ . Starting from the other side, given a category  $\mathcal{C}$  with an appropriate notion of weak equivalence and an appropriate weakened monoidal structure with unit  $U$ , consider the derived endomorphism ring spectrum  $\text{End}(U)$ . Under suitable technical conditions, we can construct Hom spectra of the appropriate homotopy type and  $\text{End}(U) = \text{Hom}(U, U)$  has an  $S$ -algebra structure under composition (or partial or  $A_\infty$   $S$ -algebra structure depending on how strictly the Hom spectra compose). The weakened monoidal structure gives us zigzags of weak equivalences between  $U' \otimes \cdots \otimes U'$  and  $U$  (where  $U'$  may be a cofibrant approximation or similar homotopical replacement), and this should (conjecturally) induce a second (partial and/or  $A_\infty$ ) structure on  $\text{End}(U)$  that satisfies an appropriate homotopy interchange property with respect to composition. Together, these can then be rectified to an  $E_2$  ring spectrum structure.

In the special case of the monoidal category of bimodules over an  $S$ -algebra, McClure and Smith [18] produced such an  $E_2$  structure, affirming the Deligne Hochschild cohomology conjecture. Since 2004, the author has been advertising the following problem, generalizing the Deligne conjecture and providing a converse.

**Problem 5.3.** *Formulate a point-set homotopy coherent  $E_{n-1}$ -monoidal structure that arises on the category of modules over an  $E_n$  ring spectrum. Prove that for a category  $\mathcal{C}$  with such a structure, under appropriate technical hypotheses, the derived endomorphism ring spectrum  $E$  of the unit is an  $E_n$ -algebra such that the induced structure on the category of  $E$ -modules is compatible with the original structure on  $\mathcal{C}$ .*

This problem has since been solved by Clark Barwick [2] and David Gepner [7]; another statement of a version of the result can be found in Lurie’s DAG-VI [14, 2.3.15].

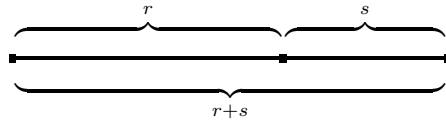
## 6. THE MOORE ALGEBRA

We close this paper with a brief note about the relationship between the left module enveloping algebra of an  $\mathfrak{A}$ -algebra and the Moore algebra. While we can make sense of the left module enveloping algebra for an algebra over an arbitrary non- $\Sigma$  operad, the Moore algebra construction is specific to algebras over  $\mathfrak{A}$ : An  $\mathfrak{A}$ -algebra  $R$  has the same relationship to its Moore algebra  $R_M$  as the based loop space of a topological space has to its Moore loop space.

We begin with the construction of the Moore algebra. For this, we let  $P = (0, \infty) \subset \mathbb{R}$  and  $\bar{P} = [0, \infty) \subset \mathbb{R}$  denote the positive real numbers and non-negative real numbers, respectively. Then as an  $S$ -module  $R_M$  is defined by the following pushout diagram.

$$\begin{array}{ccc} P_+ \wedge S & \longrightarrow & P_+ \wedge R \\ \downarrow & & \downarrow \\ \bar{P}_+ \wedge S & \longrightarrow & R_M \end{array}$$

The multiplication on  $R_M$  follows the same idea as the multiplication on the Moore loop space. We think of  $r \in P$  as specifying a length, and we use the action of  $\mathfrak{A}(2)$  on  $R$  for “concatenation”: Given  $r \in P$  and  $s \in P$ , rescaling the length  $r + s$  interval



specifies an element of  $\mathfrak{A}(2)$ , with first box length  $r/(r + s)$  and second box length  $s/(r + s)$ . We then get a map  $P \times P \rightarrow P \times \mathfrak{A}(2)$ , sending  $(r, s)$  to the length  $r + s \in P$  and the sub-intervals  $([0, r/(r + s)], [r/(r + s), 1]) \in \mathfrak{A}(2)$ . Using the  $\mathfrak{A}$ -algebra structure on  $R$ , we get the *concatenation map*

$$(P_+ \wedge R) \wedge_S (P_+ \wedge R) \cong (P \times P)_+ \wedge R \wedge_S R \longrightarrow P_+ \wedge \mathfrak{A}(2)_+ \wedge R \wedge_S R \longrightarrow P_+ \wedge R.$$

Since the map  $S \rightarrow R$  is induced by  $\mathbf{i} \in \mathfrak{A}(0)$ , the concatenation map extends to a map  $R_M \wedge R_M \rightarrow R_M$ . An easy computation shows that this provides an associative multiplication on  $R_M$ , which has unit  $S \rightarrow R_M$  induced by the inclusion of 0 in  $\bar{P}$ ,

$$S \cong \{0\}_+ \wedge S \longrightarrow \bar{P}_+ \wedge S \longrightarrow R_M.$$

We make the following definition.

**Definition 6.1.** The *Moore algebra* of  $R$  is the associative  $S$ -algebra  $R_M = (\bar{P}_+ \wedge S) \cup_{(P_+ \wedge S)} (P_+ \wedge R)$  with multiplication induced by the concatenation map as above.

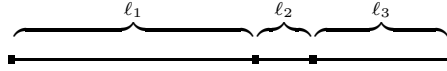
Dropping the lengths, we obtain a natural map of  $S$ -modules  $\chi: R_M \rightarrow R$ . In general the map is not a map of  $\mathfrak{A}$ -algebras, but it is a map of associative  $S$ -algebras if the  $\mathfrak{A}$ -algebra structure on  $R$  comes from an associative  $S$ -algebra structure. We also have the following analogue of Theorem 1.1.

**Proposition 6.2.** *The map  $\chi: R_M \rightarrow R$  is a homotopy equivalence of  $S$ -modules.*

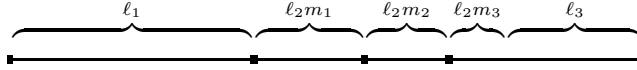
To compare the Moore algebra  $R_M$  with the left module enveloping algebra, we construct an algebra  $C = CR$  in between. Let  $E = P \times P \times \bar{P}$ ,  $\bar{E} = \bar{P} \times P \times \bar{P}$ , and define  $C$  by the following pushout diagram of  $S$ -modules.

$$\begin{array}{ccc} E_+ \wedge S & \longrightarrow & E_+ \wedge R \\ \downarrow & & \downarrow \\ \bar{E}_+ \wedge S & \longrightarrow & C \end{array}$$

The multiplication on  $C$  combines the multiplications on  $R_M$  and  $UR$ . We think of an element  $(\ell_1, \ell_2, \ell_3)$  of  $E$  as specifying an interval of length  $\ell_1 + \ell_2 + \ell_3$  together with sub-intervals of length  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  in that order.



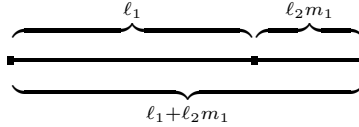
Given an element  $(m_1, m_2, m_3)$  of  $E$ , the composition  $\circ_2$  in  $\mathfrak{A}(2)$  has an analogue that associates to  $(\ell_1, \ell_2, \ell_3)$  and  $(m_1, m_2, m_3)$  an interval of length  $\ell_1 + \ell_2(m_1 + m_2 + m_3) + \ell_3$  with four sub-intervals of lengths  $\ell_1$ ,  $\ell_2 m_1$ ,  $\ell_2 m_2$ , and  $\ell_2 m_3 + \ell_3$ , in that order.



We define a map  $E \times E \rightarrow E \times \mathfrak{A}(2)$  using the map  $E \times E \rightarrow E$  that concatenates the first two sub-intervals, sending  $(\ell_1, \ell_2, \ell_3)$  and  $(m_1, m_2, m_3)$  to

$$(\ell_1 + \ell_2 m_1, \ell_2 m_2, \ell_2 m_3 + \ell_3),$$

and using the map  $E \times E \rightarrow \mathfrak{A}(2)$  that rescales the union of the first two sub-intervals to length 1,



sending  $(\ell_1, \ell_2, \ell_3)$  and  $(m_1, m_2, m_3)$  to  $([0, \ell_1/(\ell_1 + \ell_2 m_1)], [\ell_1/(\ell_1 + \ell_2 m_1), 1])$ . The map

$$(E_+ \wedge R) \wedge_S (E_+ \wedge R) \cong (E \times E)_+ \wedge R \wedge_S R \longrightarrow E_+ \wedge \mathfrak{A}(2)_+ \wedge R \wedge_S R \longrightarrow E_+ \wedge R$$

extends to a map  $C \wedge_S C \rightarrow C$  that provides the multiplication in an associative  $S$ -algebra structure. The unit is induced by the inclusion of  $(0, 1, 0)$  in  $E$ ,

$$S \cong \{(0, 1, 0)\}_+ \wedge S \longrightarrow \bar{E}_+ \wedge S \longrightarrow C.$$

We can now use  $C$  to compare  $R_M$  and  $UR$  in the category of associative  $S$ -algebras. The embedding  $P \rightarrow E$  sending  $r$  to  $(r, 1, 0)$  and the embedding  $D \rightarrow E$

$$\begin{array}{ccccc} P \times P & \longrightarrow & E \times E & \longleftarrow & D \times D \\ \downarrow & & \downarrow & & \downarrow \\ P \times \mathfrak{A}(2) & \longrightarrow & E \times \mathfrak{A}(2) & \longleftarrow & D \times \mathfrak{A}(2) \end{array}$$
$$R_M \longrightarrow CR \longleftarrow UR.$$

**Theorem 6.3.** *The maps of  $S$ -algebras  $R_M \rightarrow CR$  and  $UR \rightarrow CR$  are weak equivalences and homotopy equivalences of the underlying  $S$ -modules.*

$$R \longleftarrow R' \longrightarrow R''$$
$$\begin{array}{ccccc} R_M & \longleftarrow & R'_M & \longrightarrow & R''_M \\ & & & & \downarrow \chi \\ R & \longleftarrow & R' & \longrightarrow & R'', \end{array}$$

## REFERENCES

- [1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt. Iterated monoidal categories. *Adv. Math.*, 176(2):277–349, 2003.
- [2] Clark Barwick. Homotopy Coherent Algebra. Preprint.
- [3] Maria Basterra and Michael A. Mandell. The multiplication on  $BP$ . Preprint arXiv:1101.0023.
- [4] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 347.
- [5] Gerald Dunn. Tensor product of operads and iterated loop spaces. *J. Pure Appl. Algebra*, 50(3):237–258, 1988.
- [6] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.
- [7] David Gepner. In progress.
- [8] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.
- [9] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.
- [10] Igor Kríž and J. P. May. Operads, algebras, modules and motives. *Astérisque*, (233):iv+145pp, 1995.
- [11] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.
- [12] L. G. Lewis, Jr, J. P. May, and M. Steinberger. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
- [13] L. Gaunce Lewis, Jr. and Michael A. Mandell. Modules in monoidal model categories. *J. Pure Appl. Algebra*, 210(2):395–421, 2007.
- [14] Jacob Lurie. Derived algebraic geometry VI:  $E_k$  algebras. Preprint arXiv:0911.0018.



- [15] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc.* (3), 82(2):441–512, 2001.
- [16] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.
- [17] J. P. May. Pairings of categories and spectra. *J. Pure Appl. Algebra*, 19:299–346, 1980.
- [18] James E. McClure and Jeffrey H. Smith. A solution of Deligne’s Hochschild cohomology conjecture. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 153–193. Amer. Math. Soc., Providence, RI, 2002.

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